PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 6, June 1995

A COVERING COCYCLE WHICH DOES NOT GROW LINEARLY

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ABSTRACT. A cocycle $h : X \times Z^m \to R^n$ of a Z^m action on a compact metric space, provides an R^n suspension flow (analogous to a flow under a function) on a space X_h which may not be Hausdorff or even T_1. Linear growth of h guarantees that X_h is a Hausdorff space; when $m = n$, linear growth is a consequence of X_h being Hausdorff and a covering condition. This paper contains the construction of a cocycle $h : X \times Z \to R^2$ which does not grow linearly yet produces a locally compact Hausdorff space with the covering condition. The Z action used in the construction is a substitution minimal set.

1. INTRODUCTION

Let X be a compact metric space, and let Z^m act as a group of commuting homeomorphisms on X. That is, we have a flow (X, Z^m). For $a \in Z^m$, we denote the action of a on $x \in X$ by ax. A cocycle for the flow (X, Z^m) is a continuous map $h : X \times Z^m \to R^n$ satisfying the cocycle equation: for any $a, b \in Z^m$ and $x \in X$,

$$h(x, a+b) = h(x, a) + h(ax, b).$$

A cocycle $h : X \times Z^m \to R^n$ can be used to construct the suspension (X_h, R^n) of the flow (X, Z^m). This is done as follows: we have a Z^m action on $X \times R^n$ given by

$$T_a(x, w) = (ax, w - h(x, a))$$

for $a \in Z^m$, $x \in X$, $w \in R^n$. Because h is a cocycle, it is easily checked that $T_a \circ T_b = T_{a+b}$ and hence $a \to T_a$ defines a Z^m action on $X \times R^n$. We also have a natural R^n action on $X \times R^n$ via

$$((x, v), w) \to (x, v + w)$$

for $x \in X$, $v, w \in R^n$. These two actions commute and so the R^n flow on $X \times R^n$ gives an R^n action on X_h, where X_h is the quotient space of $X \times R^n$ modulo the Z^m action on it. When $m = n = 1$, this construction yields the familiar flow under a function.

Received by the editors April 4, 1994; originally communicated to the *Proceedings of the AMS* by Linda Keen.

1991 *Mathematics Subject Classification*. Primary 58F25; Secondary 28D10, 54H20.

Research was supported in part by NSF Grant #01524638.

0002-9939/95 $1.00 + $.25 per page

In the hope of being able to use suspensions to create interesting R^n flows, we would like to insist that X_h is Hausdorff and (X_h, R^n) is minimal. It is not hard to check that (X_h, R^n) will be minimal if and only if (X, Z^m) is minimal.

The relationship between the properties of h and the topological properties of the resulting suspension (X_h, R^n) has been studied by Furstenberg, Keynes, Markley, and Sears in [3].

Definition 1.1. A cocycle h is covering if

(i) X_h is a Hausdorff space,
(ii) the projection $\pi : X \times R^n \to X_h$ is a local homeomorphism.

When h is covering, X_h is locally compact and metric. One of the goals of [3] was to gain a basic understanding of covering cocycles. Some of the results are included here under the assumption that the Z^m action on X has a free dense orbit.

Theorem 1.1. *A cocycle $h : X \times Z^m \to R^n$ is covering if and only if $|h(x, a)| \to \infty$ uniformly in x as $|a| \to \infty$.*

(The norm we will use is $|v| = \sum |v_i|$.)

Theorem 1.2. *Let $h : X \times Z^m \to R^n$ be a covering cocycle. Then $n \geq m$, and the space X_h is compact if and only if $n = m$.*

The key result needed to show that $m = n$ when X_h is compact is:

Theorem 1.3. *If h is a covering cocycle and X_h is compact, then there exist constants C and D such that*

$$C|a| \leq |h(x, a)|$$

whenever $|a| \geq D$.

In fact, as a consequence of the cocycle equation, given any cocycle $h : X \times Z^m \to R^n$, there exists a constant $C' > 0$ such that $|h(x, a)| \leq C'|a|$ for all $a \in Z^m$. Thus, for h as in Theorem 1.3,

$$C|a| \leq |h(x, a)| \leq C'|a|$$

when $|a|$ is sufficiently large.

These results lead to the question of whether the covering cocycles are precisely the cocycles which grow linearly in the sense of Theorem 1.3. Clearly, when $n = m$ the answer is yes. In what follows we will show that when $n \neq m$, covering and linear growth are not equivalent notions. We will construct a covering cocycle on a minimal flow which does not grow linearly.

Our example is obtained using a substitution minimal flow $(\overline{\mathscr{O}(\omega)}, \sigma)$ and a cocycle $h : \overline{\mathscr{O}(\omega)} \times Z \to R^2$. In section 2 we discuss some general properties of substitution minimal flows and cocycles on such flows. In section 3 we construct the specific substitution minimal flow and cocycle h on it which is covering but does not grow linearly.

2. Substitution minimal flows

Let (X, T) be a flow (i.e., $T: X \to X$ is a homeomorphism of X onto X), and suppose that $\omega \in X$ has a free dense orbit. If $h: X \times Z \to R^n$ is a cocycle for the flow (X, T), we may characterize when h is covering in terms of ω:

Proposition 2.1. *Let (X, T), $\omega \in X$, and h be as described. Then h is covering if and only if given $R > 0$, there exists $r > 0$ such that $|a - b| > r$ implies $|h(\omega, a) - h(\omega, b)| > R$.*

Proof. First suppose h is covering and let $R > 0$ be given. We can choose $r > 0$ such that $|a| > r$ implies $|h(x, a)| > R$ for all $x \in X$. In particular, using the cocycle equation,

$$|h(\omega, a) - h(\omega, b)| = |h(T^b(\omega), a - b)| > R$$

when $|a - b| > r$.

Conversely, let $R > 0$ be given and choose $r > 0$ as described. Now suppose $x \in X$ and $a \in Z$ with $|a| > r$. For some $\{n_i\}_{i=0}^{\infty} \subseteq Z$, $T^{n_i}(\omega) \to x$ and

$$\begin{aligned} |h(x, a)| &= \lim_{i \to \infty} |h(T^{n_i}(\omega), a)| \\ &= \lim_{i \to \infty} |h(\omega, a + n_i) - h(\omega, n_i)| \\ &\geq R. \end{aligned}$$

Thus, $|h(x, a)| \to \infty$ uniformly in $x \in X$ as $|a| \to \infty$. □

We also observe that for (X, T) a flow, there is a one-to-one correspondence between $\mathscr{C} = \{h: X \times Z \to R^n | h \text{ is a cocycle}\}$ and $C(X, R^n) = \{f: X \to R^n | f \text{ is continuous}\}$. For $f \in C(X, R^n)$ we obtain $h_f \in \mathscr{C}$ as follows:

$$h_f(x, n) = \begin{cases} \sum_{i=0}^{n-1} f(T^i x), & n > 0, \\ 0, & n = 0, \\ -\sum_{i=1}^{-n} f(T^{-i} x), & n < 0. \end{cases}$$

Conversely, if $h \in \mathscr{C}$, then $h(\cdot, 1): X \to R^n$ is in $C(X, R^n)$.

In particular, for $S = \{1, 2, \dots, m\}$ a finite collection of symbols and (S^Z, σ) the usual full shift on m symbols, (S^Z, σ) is a flow. By choosing a non-periodic point $\omega \in S^Z$, we can construct $(\overline{\mathscr{O}(\omega)}, \sigma) = (X, \sigma)$ where $\overline{\mathscr{O}(\omega)} = \overline{\{\sigma^k(\omega): k \in Z\}}$. Then (X, σ) is a flow with $\omega \in X$ having a free dense orbit. We obtain a collection of cocycles on (X, σ) by considering maps from S into R^n. If $f: S \to R^n$, then for $\tilde{f}: X \to R^n$ via $\tilde{f}(x) = f(x_0)$, $\tilde{f}$ is in $C(X, R^n)$ and gives $h = h_{\tilde{f}}$ in $\mathscr{C}$ as described. (We may extend f to finite blocks of symbols by $f(s_1 \dots s_k) = \sum_{i=1}^{k} f(s_i)$. Then $h(x, n) = f(x[0, n-1])$ for $n > 0$ and $h(x, n) = -f(x[n, -1])$ for $n < 0$.) Under these circumstances and the additional assumption that (X, σ) is uniquely ergodic, the following proposition illustrates that the linear growth of h and the values taken on by f are closely connected.

Proposition 2.2. *Let (X, σ), f, and h be as described. Suppose also that (X, σ) is uniquely ergodic with unique invariant measure ϕ. Then the following are equivalent:*

(1) *There exists* $r, R, R' > 0$ *such that* $|a| > r$ *implies*

$$R'|a| \leq |h(x, a)| \leq R|a|$$

for all $x \in X$

(2) *There exists* $r, R, R' > 0$ *such that* $|a-b| > r$ *implies*

$$R'|a-b| \leq |h(\omega, a) - h(\omega, b)| \leq R|a-b|$$

(3) $|\sum_{s\in S} f(s)\phi(s)| > 0$.

Proof. It is clear that (1) implies (2) since

$$|h(\omega, a) - h(\omega, b)| = |h(\sigma^b(\omega), a-b)|.$$

Suppose that (2) holds. Let $x \in X$ with $\sigma^{n_i}(\omega) \to x$. Then

$$|h(x, a)| = \lim_{i\to\infty} |h(\sigma^{n_i}(\omega), a)| = \lim_{i\to\infty} |h(\omega, a+n_i) - h(\omega, n_i)|,$$

and it is clear that (2) implies (1).

Now suppose that $a, b \in Z$, $a < b$. We see

$$|h(\omega, a) - h(\omega, b)| = \left|\sum_{s\in S} f(s) N_s(\omega[a, b-1])\right|,$$

where $N_s(\omega[a, b-1])$ denotes the number of occurrences of the symbol s in the block $\omega[a, b-1]$. In particular, for any $a \in Z$, by the unique ergodicity of (X, σ)

$$\lim_{m\to\infty}\left|\frac{h(\omega, a) - h(\omega, a+m)}{m}\right| = \lim_{m\to\infty}\left|\sum_{s\in S} f(s)\frac{N_s(\omega[a, a+m-1])}{m}\right|$$
$$= \left|\int \tilde{f}\, d\phi\right| = \left|\sum_{s\in S} f(s)\phi(s)\right|.$$

Since this limit is uniform in $a \in Z$, it is apparent that (2) holds if and only if (3) holds. □

Notice that because the constant R in (1) exists for any cocycle, if (3) fails to hold then (1) fails because R' does not exist.

With these results in mind, we turn to substituting minimal flows as a way of constructing uniquely ergodic, minimal flows (X, σ) with $X = \overline{\mathscr{O}(\omega)} \subseteq S^Z$. A substitution is a map sending symbols from a finite set into finite strings of these symbols, called blocks. A thorough overview of substitutions can be found in [1] and [8]. For completeness, we include some of the notation and results here.

For $S = \{1, 2, \ldots, m\}$ as before, let S^n denote the collection of blocks of length n. Then $\theta : S \to S^n$ is a substitution of constant length. We associate an $m \times m$ matrix, M_θ, with θ via

$$M_\theta = (N_i(\theta(j)))_{1\leq i, j\leq m}$$

where $N_i(\theta(j))$ is the number of occurrences of the symbol i in the block $\theta(j)$. We say θ is primitive if the matrix M_θ is primitive in the usual sense.

Let θ be a primitive substitution. We say a pair of symbols $p, q \in S$ is a recurrent pair if

(i) $\theta(q)$ begins with q and $\theta(p)$ ends with p,
(ii) the block pq occurs in $\theta^k(q)$ for some $k \geq 1$.

A recurrent pair of symbols determines an element of S^Z, denoted $\omega^{pq} = \omega$, by

$$\omega[-n^k, n^k - 1] = \theta^k(pq)$$

where $\omega[i, i+j] = \omega_i \omega_{i+1} \dots \omega_{i+j}$. Notice that $\theta(\omega) = \omega$. Then $\overline{\mathscr{O}(\omega)} = \overline{\{\sigma^k(\omega) : k \in Z\}}$ is an invariant subset of S^Z. In fact we can say more:

Theorem 2.1. *If θ is a substitution with a recurrent pair pq and $\omega = \omega^{pq} \in S^Z$, then $(\overline{\mathscr{O}(\omega)}, \sigma)$ is minimal and uniquely ergodic.*

The minimality of $(\overline{\mathscr{O}(\omega)}, \sigma)$ was first observed by Gottschalk [4] and the unique ergodicity by Klein [5]. Others, including Dekking [1] and Michel [7], have extended these results to more general situations than those discussed here.

If ϕ is the unique invariant measure on $\overline{\mathscr{O}(\omega)}$, then $(\phi(s))_{s \in S}$ is the normalization of the strictly positive right eigenvector associated with the Perron-Frobenius eigenvalue of M_θ [7].

3. A covering cocycle which does not grow linearly

Let $S = \{a, b, c, d, \hat{a}, \hat{b}, \hat{c}, \hat{d}\}$ and let $\theta : S \to S^4$ as follows:

$$\theta(a) = abca, \quad \theta(b) = bdab, \quad \theta(c) = ca\hat{d}c, \quad \theta(d) = d\hat{c}bd,$$

and $\theta(\hat{s}) = \hat{s}\hat{s}_1\hat{s}_2\hat{s}$ for $\theta(s) = ss_1s_2s$ and $\hat{\hat{s}} = s$.

We see that

$$M_\theta = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 \end{pmatrix},$$

and it is easily verified that M_θ is primitive and that ca is a recurrent pair. Thus, for $\omega = \omega^{ca}$, $(\overline{\mathscr{O}(\omega)}, \sigma) = (X_\omega, \sigma)$ is minimal and uniquely ergodic. Let ϕ be the unique invariant measure on (X_ω, σ). Since the Perron-Frobenius eigenvalue of M_θ is 4 and the corresponding right eigenvector is $(1, 1, \dots, 1)$, $\phi(s) = 1/8$ for all $s \in S$.

Let $f : S \to R^2$ by

$$f(a) = (\sqrt{2}, 0), \qquad f(b) = (1, 1),$$
$$f(c) = (1, -1), \qquad f(d) = (0, \sqrt{2})$$

and $f(\hat{s}) = -f(s)$. Then, as described in the previous section, we have $\tilde{f} \in C(X_\omega, R^2)$ and $h = h_{\tilde{f}} \in \mathscr{C} = \{ g : X_\omega \times Z \to R^2 | g \text{ is a cocycle} \}$. By Proposition 2.2, h does not grow linearly since $\left| \sum_{s \in S} f(s)\phi(s) \right| = 0$. Also, if we let $\tilde{f}(x) = (\tilde{f}_1(x), \tilde{f}_2(x))$, then $\int_{X_\omega} \tilde{f}_i d\phi = 0$ for $i = 1, 2$ and so, by Theorem 1.12 in [3], neither $\tilde{f}_1$ nor $\tilde{f}_2$ would determine a covering cocycle on X_ω. In particular, trying to build a flow under either $\tilde{f}_1$ or $\tilde{f}_2$ would be fruitless. However, in what follows we will show that taken in tandem, they produce h, a covering cocycle with values in R^2.

We first note that for

$$L = \begin{pmatrix} 2+\sqrt{2} & 0 \\ 0 & 2+\sqrt{2} \end{pmatrix}$$

$Lf = f\theta$ and, by induction, $L^n f = f\theta^n$. Denote $2+\sqrt{2}$ by α. (The substitution θ and the matrix L are variants of a substitution and matrix used by Dekking in [2] to create a continuous nowhere differentiable curve, the von Koch curve. Dekking uses the substitution to construct compact subsets $\{A_n\}_{n=1}^{\infty}$ of R^2; his compact sets, A_n, resemble the image of the first 4^n integers (connected with line segments) under our $h(\omega, \cdot)$. Dekking then uses L^{-1} as a scaling transformation to obtain the von Koch curve as the limit of the sets $L^{-n}A_n$.)

To show that h is covering, by Proposition 2.1 it will suffice to show that given $R > 0$ there exists $r > 0$ so that when $|n - m| > r$ for $n, m \in Z$ (without loss of generality $n < m$), then

$$|h(\omega, n) - h(\omega, m)| = |f(\omega[n, m-1])| > R.$$

The following three lemmas will be used.

Lemma 3.1. *Let W be a block appearing in ω, and let $N(W)$ denote the number of symbols in W. Then for all $n \geq 0$,*

(i) $N(W) = 1$ *implies* $\sqrt{2}\alpha^n \leq |f\theta^n(W)| \leq 2\alpha^n$,
(ii) $N(W) = 2$ *implies* $2\alpha^n \leq |f\theta^n(W)| \leq \alpha^{n+1}$,
(iii) $N(W) = 3$ *implies* $\alpha^{n+1} \leq |f\theta^n(W)| \leq \sqrt{2}\alpha^{n+1}$,
(iv) $N(W) = 4$ *implies* $\sqrt{2}\alpha^{n+1} \leq |f\theta^n(W)| \leq 2\alpha^{n+1}$,
(v) $N(W) = 5$ *implies* $\sqrt{2}\alpha^{n+1} \leq |f\theta^n(W)|$.

Proof. We observe that it suffices to show that the inequalities hold for $n = 0$ since

$$|f\theta^n(W)| = |L^n f(W)| = \alpha^n |f(W)|.$$

To show that the lemma holds for $n = 0$, we list all the blocks of length 5 or less appearing in ω and we apply f to all of these blocks.

This is not as tedious as it might seem. There are several simplifying assumptions we can make. First, it can be shown that each symbol can only be followed by two others. For example, a can be followed by b and $\hat{d}$. Secondly, if $s_1 \dots s_n$ appears in ω for $1 \leq n \leq 5$, then we must be able to find $t_1 t_2$ appearing in ω with $s_1 \dots s_n$ a subblock of $\theta(t_1 t_2)$. These two observations considerably shorten the list of possible words of length 5 or less appearing in ω. Also the fact that $|f(s_1 \dots s_n)| = |-f(\hat{s}_1 \dots \hat{s}_n)|$ reduces the necessary calculations further. These calculations can be found in Table 1 of [6]. □

The remaining two lemmas give approximations for values of f on blocks of the form

$$\theta^{n-1}(s_1 \dots s_i)\theta^n(t_1 \dots t_k)\theta^{n-1}(u_1 \dots u_j),$$

$0 \le i, j \le 3$, $1 \le k \le 4$, when they occur in ω in a particular location. The word "occurs" will be given a special technical meaning; specifically, when we say a block of this form *occurs* in ω, we mean it appears in the following way: $\theta(s_i)$ ends in $s_1 \dots s_i$, $\theta(u_1)$ begins with $u_1 \dots u_j$, and

$$\omega[0, p \cdot 4^n - 1] = \theta^n(a) \dots \theta^n(s_i)\theta^n(t_1 \dots t_k)\theta^n(u_1)$$

where $\omega[0, p-1]$ begins with a and ends with $s_i t_1 \dots t_k u_1$ or that

$$\omega[-p \cdot 4^n, -1] = \theta^n(s_i)\theta^n(t_1 \dots t_k)\theta^n(u_1) \dots \theta^n(c)$$

where $\omega[-p, -1]$ begins with $s_i t_1 \dots t_k u_1$ and ends with c. This distinction is important because not every appearance of a block of the above form is an occurrence.

Lemma 3.2. *Suppose* $0 \le i \le 3$, $2 \le k \le 4$.

(1) *If* $s_1 \dots s_i \theta(t_1 \dots t_k)$ *occurs in* ω, *then*

$$|f(s_1 \dots s_i \theta(t_1 \dots t_k))| \ge |f(s_1 \dots s_i)| + |f(\theta t_1)|.$$

(2) *If* $\theta(t_1 \dots t_k) s_1 \dots s_i$ *occurs in* ω, *then*

$$|f(\theta(t_1 \dots t_k) s_1 \dots s_i)| \ge |f(\theta t_1)| + |f(s_1 \dots s_i)|.$$

Proof. If $i = 0$, then the result follows from Lemma 3.1. The case of $k = 2$ is handled by calculating $|f(s_1 \dots s_i \theta(t_1 \dots t_k))|$, $|f(s_1 \dots s_i)| + |f(\theta t_1)|$, and $|f(\theta(t_1 \dots t_k) s_1 \dots s_i)|$ on all possible blocks of length three occurring in ω. These calculations are contained in Table 2 of [6].

Otherwise, $k > 2$. We will show that (1) holds. The proof of (2) is similar.

First, suppose that $i = 1$. Then, using Lemma 3.1 twice,

$$\begin{aligned} |f(s_1\theta(t_1 \dots t_k))| &\ge \big| |f(\theta(t_1 \dots t_k))| - |f(s_1)| \big| \ge \alpha^2 - 2 \\ &\ge 2\alpha + 2 \ge |f(\theta t_1)| + |f(s_1)|. \end{aligned}$$

Next suppose that $i = 2$ and suppose $\theta(s_2) = s_2 s_3 s_1 s_2$. Then

$$\begin{aligned} |f(s_1 s_2 \theta(t_1 \dots t_k))| &\ge \big| |f(\theta(s_2 t_1 \dots t_k))| - |f(s_2 s_3)| \big| \\ &\ge \sqrt{2}\alpha^2 - \alpha \ge 2\alpha + \alpha \\ &\ge |f(\theta t_1)| + |f(s_1 s_2)|. \end{aligned}$$

A similar argument holds for $i = 3$.

Lemma 3.3. *Suppose* $s_1 \dots s_i \theta(t_1 \dots t_k) u_1 \dots u_j$ *occurs in* ω, *with* $0 \le i, j \le 3$ *and* $1 \le k \le 4$. *Then, for all* $n \in N$,

$$\left| f[\theta^{n-1}(s_1 \dots s_i)\theta^n(t_1 \dots t_k)\theta^{n-1}(u_1 \dots u_j)] \right| \ge \sqrt{2}\alpha^n.$$

Proof. We observe that the proof reduces to showing that the inequality holds for $n = 1$, because for $n > 1$, we have the following easy inductive step:

$$\begin{aligned}
&|\, f(\theta^{n-1}(s_1 \dots s_i)\theta^n(t_1 \dots t_k)\theta^{n-1}(u_1 \dots u_j))\,| \\
&\quad = |\, Lf(\theta^{n-2}(s_1 \dots s_i)\theta^{n-1}(t_1 \dots t_k)\theta^{n-2}(u_1 \dots u_j))\,| \\
&\quad = \alpha|\, f(\theta^{n-2}(s_1 \dots s_i)\theta^{n-1}(t_1 \dots t_k)\theta^{n-2}(u_1 \dots u_j))\,| \\
&\quad \geq \alpha(\sqrt{2}\alpha^{n-1}) \\
&\quad = \sqrt{2}\alpha^n.
\end{aligned}$$

First suppose that $k = 1$. Lemma 3.1 includes $i = j = 0$. If $j = 3$ and $i > 1$ then, using Lemma 3.1 and Lemma 3.2,

$$\begin{aligned}
|\, f(s_1 \dots s_i\theta(t_1)u_1u_2u_3)\,| &\geq |\,|f(s_1 \dots s_i\theta(t_1u_1))| - |f(u_1)|\,| \\
&\geq |\, f(\theta t_1)\,| + |\, f(s_1 \dots s_i)\,| - |\, f(u_1)\,| \\
&\geq |\, f(\theta t_1)\,| \\
&\geq \sqrt{2}\alpha.
\end{aligned}$$

The case of $j > 1$ and $i = 3$ is similar. The remaining cases for $k = 1$ are handled in Tables 3 and 4 of [6].

Otherwise, $k \geq 2$. Suppose $|f(s_1 \dots s_i)| \geq |f(u_1 \dots u_j)|$. Then, using Lemma 3.2,

$$\begin{aligned}
|\, f(s_1 \dots s_i\theta(t_1 \dots t_k)u_1 \dots u_j))\,| &\geq |\,|f(s_1 \dots s_i\theta(t_1 \dots t_k))| - |f(u_1 \dots u_j)|\,| \\
&\geq |\, f(\theta t_1)\,| + |\, f(s_1 \dots s_i)\,| - |\, f(u_1 \dots u_j)\,| \\
&\geq |\, f(\theta t_1)\,| \\
&\geq \sqrt{2}\alpha.
\end{aligned}$$

Similarly if $|f(u_1 \dots u_j)| \geq |f(s_1 \dots s_i)|$.

We are now ready to show that h is covering.

Theorem 3.1. *There exists* $c > 0$ *such that when* $4^n < |a - b| \leq 4^{n+1}$ *then*

$$|\, h(\omega, a) - h(\omega, b)\,| \geq c\alpha^{n-1}$$

for all $n \in N$ *and* h *is a covering cocycle.*

Proof. Assuming $a < b$, first note that if

$$W = s_{0_1} \dots s_{0_{i_0}}\theta(s_{1_1} \dots s_{1_{i_1}})\theta^2(s_{2_1} \dots s_{2_{i_2}}) \dots \theta^{n-2}(s_{(n-2)_1} \dots s_{(n-2)_{i_{(n-2)}}})$$

appears in ω with $0 \leq i_j \leq 3$ and $0 \leq j \leq n-2$, then, using Lemma 3.1, we have

$$\begin{aligned}
|\, f(W)\,| &\leq \sum_{j=0}^{n-2} |\, f(\theta^j(s_{j_1} \dots s_{j_{i_j}})\,| \leq \sum_{j=0}^{n-2} \sqrt{2}\alpha^{j+1} \\
&= \sqrt{2}\alpha\left(\frac{1-\alpha^{n-1}}{1-\alpha}\right) = \sqrt{2}\alpha\left(\frac{\alpha^{n-1}-1}{\alpha-1}\right) \leq \frac{\sqrt{2}\alpha^n}{\alpha-1}\quad.
\end{aligned}$$

We consider two possible cases.

Case 1. A $\theta^n(s)$ block occurs in $\omega[a, b-1]$ for some $s \in S$. (Notice at most four such blocks can occur in $\omega[a, b-1]$.) So

$$\begin{aligned}\omega[a, b-1] = {} & s_{0_1} \dots s_{0_{i_0}} \theta(s_{1_1} \dots s_{1_{i_1}}) \\ & \dots \theta^{n-2}(s_{(n-2)_1} \dots s_{(n-2)_{i_{(n-2)}}}) \theta^{n-1}(s_1 \dots s_i) \theta^n(t_1 \dots t_k) \theta^{n-1}(u_1 \dots u_j) \\ & \dots u_{0_1} \dots u_{0_{j_0}}\end{aligned}$$

with $\theta^{n-1}(s_1 \dots s_i)\theta^n(t_1 \dots t_k)\theta^{n-1}(u_1 \dots u_j)$ occurring in ω and where $0 \le i_l$, i, $j_l, j \le 3$, $1 \le k \le 4$, and $0 \le l \le n-2$. Applying the previous estimate and Lemma 3.3, we obtain

$$\begin{aligned}|h(\omega, a) - h(\omega, b)| &= |f(\omega[a, b-1])| \\ &\ge \left| |f(\theta^{n-1}(s_1 \dots s_i)\theta^n(t_1 \dots t_k)\theta^{n-1}(u_1 \dots u_j))| - 2\left(\frac{\sqrt{2}\alpha^n}{\alpha - 1}\right) \right| \\ &\ge \left(\sqrt{2} - \frac{2\sqrt{2}}{\alpha - 1}\right)\alpha^n.\end{aligned}$$

Now set $c = \left(\sqrt{2} - \frac{2\sqrt{2}}{\alpha-1}\right) \approx .2426 > 0$.

Case 2. No $\theta^n(s)$ blocks occur in $\omega[a, b-1]$. Then at least three and at most six $\theta^{n-1}(s)$ blocks occur in $\omega[a, b-1]$. If three or four $\theta^{n-1}(s)$ blocks occur, then the argument used in Case 1 with $n-1$ gives $|h(\omega, a) - h(\omega, b)| \ge c\alpha^{n-1}$.

Otherwise, five or six $\theta^{n-1}(s)$ blocks occur in $\omega[a, b-1]$. Say $\theta^{n-1}(s_1 \dots s_i)$ occurs in $\omega[a, b-1]$ for $5 \le i \le 6$. But then $s_1 \dots s_i$ appears in $\theta(t_1 t_2)$ and $\omega[a, b-1]$ appears in $\theta^n(t_1 t_2)$. In this case, using Lemma 3.1,

$$\begin{aligned}|h(\omega, a) - h(\omega, b)| &= |f(\omega[a, b-1])| \\ &\ge \left| |f(\theta^n(t_1 t_2))| - 3 \max_{s \in S} |f(\theta^{n-1}(s))| \right| \\ &\ge \left|2\alpha^n - 3(2\alpha^{n-1})\right| \\ &\ge (2\alpha - 6)\alpha^{n-1} \\ &\approx (.8284)\alpha^{n-1} \\ &\ge c\alpha^{n-1}\end{aligned}$$

as desired.

Now we can summarize the properties of the cocycle we have constructed:

Theorem 3.2. *There exists a uniquely ergodic substitution minimal flow* (X_ω, σ) *and a cocycle* $h : X_\omega \times Z \to R^2$ *such that*

(i) *the cocycle* h *does not grow linearly,*

(ii) *the cocycles* $h_i : X_\omega \times Z \to R$ *are not covering for* $i = 1, 2$ *where*

$$h(x, a) = (h_1(x, a), h_2(x, a)),$$

(iii) *the cocycle* h *is covering.*

Proof. Use Proposition 2.2, Theorem 1.12 of [3], Proposition 2.1, and Theorem 3.1.

Finally, it follows from Proposition 2.2 that some arbitrarily small perturbations of the h we have constructed will have linear growth. However, we do not know how such perturbations effect the structure of the locally compact Hausdorff space X_h and the R^2 action on it.

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